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A BEAM ON A WIEGHARDT-TYPE ELASTIC FOUNDATION

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Abstract—It is assumed that the differential equation of the deflexion curve of the beam (6), in which the effect of shear on the curvature of the beam has been taken into account, is valid for a beam on an elastic foundation. According to Wieghardt, the relationship (4) with kernel function (5) is assumed between the deflexion and the foundation pressure. On the basis of these assumptions, the differential equation of deflexion curve (14) is then obtained. This differential equation has been employed in examination of two cases of loading a beam of finite length. The corresponding results for an infinitely long beam are also derived.

NOTATION

A	cross-sectional area of the beam
B_1, B_2, B_3, B_4	integration constants
C	modulus of foundation according to equation (1)
E	modulus of elasticity
F	force
G	modulus of rigidity
I	moment of inertia of the cross-section
M	bending moment
N_1, N_2, N_3	abbreviations according to equations (32), (34) and (36)
Q	shear force
R, R_0, R_1, R_2	concentrated foundation pressures
a, b, d	abbreviations according to equation (13)
c	constant in the kernel function (5)
k	constant in the kernel function (5)
l	length of the beam
p	foundation pressure per unit length
q	continuous load per unit length
v	deflexion of the beam and/or of the foundation
x, y	co-ordinates
$\alpha, \beta, \gamma, \delta, \varepsilon$	notations according to equations (17), (19) and (21)
β_0	abbreviation according to equation (13)
$\varkappa$	coefficient in equation (6)

1. INTRODUCTION

CONSIDERATION is given here to a prismatic beam lying on an elastic isotropic foundation (Fig. 1). Right-angled co-ordinate axes are chosen so that the x-axis coincides with the axis of the unloaded beam. The positive direction of the deflexion $v(x)$ is downwards, and it is assumed that the deflexion of the axis of the beam equals the deflexion of the foundation. The continuous loading which affects the upper surface of the beam per unit length is denoted by $q(x)$ and the pressure per unit length produced by the lower surface of the beam on the foundation is denoted by $p(x)$. It is assumed that the beam remains on the foundation for its whole length and the horizontal forces which appear on the contacting surface of the beam and the foundation are ignored. As a rule it is assumed, in accord with Winkler [1] and Zimmermann [2], that the foundation pressure is proportional, at every point, to the deflexion at the same point, or that

$$p(x) = Cv(x). \tag{1}$$

The modulus of foundation C has the dimension of the modulus of elasticity. Regarding the other relationship connecting the unknown functions $p(x)$ and $v(x)$, Winkler takes into account the differential equation of the deflexion curve of the beam in the engineering theory of bending

$$EI\frac{\mathrm{d}^4v(x)}{\mathrm{d}x^4} = q(x)-p(x) \tag{2}$$

where E is the modulus of elasticity of the material of the beam and I the moment of inertia of the cross-section of the beam. From equations (1) and (2), follows the differential equation of the deflexion curve of the beam on an elastic foundation

$$EI\frac{\mathrm{d}^4v(x)}{\mathrm{d}x^4}+Cv(x) = q(x). \tag{3}$$

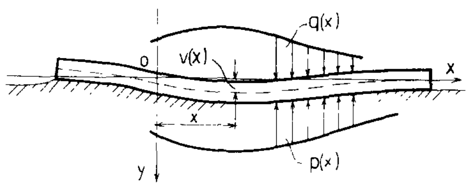

FIG. 1

This elementary theory by Winkler has been subjected to severe criticism by Wieghardt [3] as regards both assumption (1) and the use of differential equation (2). He remarks that hypothesis (1) is defective, not by reason of the proportionality between the pressure and deflexion, which can be considered valid at small deflexions, but because the deflexion at one point depends only upon the pressure at that point, and not at all upon the pressure in the surroundings of the point. The inaccuracy of hypothesis (1) is most clearly revealed at the ends of a finite beam, where according to equation (1) the foundation surface deflexion becomes discontinuous, which is in contradiction to observations made.

Wieghardt replaces the hypothesis by a more rational one

$$v(x) = \int_l K(|x-\xi|)p(\xi)\,\mathrm{d}\xi \tag{4}$$

where the integration has to be taken over the length of beam l. As the kernel function $K(|x-\xi|)$, Wieghardt chooses the exponential function

$$K(|x-\xi|) = c\,\mathrm{e}^{-k|x-\xi|} \tag{5}$$

chiefly with a view to avoiding mathematical difficulties, and according to the tests made by Föppl [4], it describes, approximately, the distribution of the pressure in the soil. The constants c and k depend upon the properties of the foundation.

Schiel [5] has pointed out that a mechanical model of the foundation, characterized by the kernel function (5), is a heavy liquid with surface tension. He has examined a beam resting on the surface of such a liquid, and has also arrived at an exponential kernel function. From the surface tension, it follows that if the curve which represents a section of the surface has a vertex, i.e. the slope of the curve has a point of discontinuity, a concentrated pressure appears. Wieghardt states that the foundation pressure should be a continuous function and cannot accept concentrated pressures. He implies that these irregularities are caused by the inaccuracy of the differential equation (2) and endeavours to avoid the contradiction, rejecting differential equation (2), and employing the Airy stress-function.

In the following it is accepted the appearance of concentrated foundation pressures at points where the slope of the curve which represents the surface of the foundation has a point of discontinuity, and it is considered that this property arises from the characteristic of the foundation, determined by the kernel function (5).

We also regard the differential equation of the deflexion curve of the beam as valid, but in a form such that the effect of the shear stresses on the curvature of the beam has been taken into account [6–9]

$$\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = -\frac{M}{EI} + \frac{\varkappa}{GA}\frac{\mathrm{d}Q}{\mathrm{d}x}. \tag{6}$$

In equation (6), M and Q are, respectively, the bending moment and the shear force of the beam, G is the modulus of rigidity of the material of the beam, A the area of the cross-section of the beam, $\varkappa$ a numerical coefficient the value of which depends on the form of the cross-section and on Poisson's ratio of the material of the beam.

Equation (4) includes the elementary hypothesis (1) as a special case, as is observable on taking for the kernel function

$$K(|x-\xi|) = \frac{1}{C}\delta(|x-\xi|)$$

where δ is the Dirac delta-function defined by equations $\delta(t) = 0, t \neq 0$, and $\int_{-\infty}^{+\infty} \delta(t)\,\mathrm{d}t = 1$. With reference to the exponential kernel function (5), it can be shown that we must have

$$\frac{k}{2c} = C \qquad \text{when} \quad k \to \infty$$

if correspondence is to be achieved between the coefficients in assumptions (1) and (5).

2. DIFFERENTIAL EQUATION OF THE DEFLECTION CURVE OF THE BEAM

It is supposed that the beam is subjected to a distributed load $q(x)$ and a single concentrated force F (Fig. 2), and the origin of the co-ordinate axes is taken to be at the point of application of the force.

From the expression of the curvature of the beam (6), it can be deduced that the slope of the deflexion curve has the same points of discontinuity as shear force Q of the beam, and consequently the points of application of concentrated loads. It can also be expected that at the ends of the beam the deflexion curve has vertices, thus giving rise to concentrated foundation pressures.

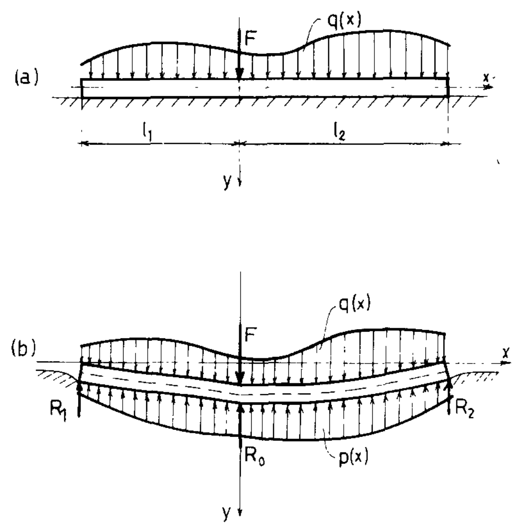

FIG. 2

Thus the expression of the deflexion curve of the beam in Fig. 2 can be presented in the form

$$v(x) = c\int_{-l_1}^{x} e^{-k(x-\xi)}p(\xi)\,d\xi + c\int_{x}^{l_2} e^{-k(\xi-x)}p(\xi)\,d\xi + cR_1\,e^{-k(x+l_1)} + cR_0\,e^{-k|x|} + cR_2\,e^{-k(l_2-x)} \qquad (-l_1 \leqq x \leqq l_2) \tag{7}$$

where R_1, R_2 and R_0 are the concentrated foundation pressures at the ends of the beam and at the origin. Outside the beam, the deflexion curve of the surface is, say for $x \geqq l_2$

$$v(x) = c\int_{-l_1}^{l_2} e^{-k(x-\xi)}p(\xi)\,d\xi + cR_1\,e^{-k(x+l_1)} + cR_0\,e^{-kx} + cR_2\,e^{-k(x-l_2)}. \tag{8}$$

If expression (7) is differentiated twice, then

$$\frac{dv}{dx} = -ck\int_{-l_1}^{x} e^{-k(x-\xi)}p(\xi)\,d\xi + ck\int_{x}^{l_2} e^{-k(\xi-x)}p(\xi)\,d\xi - ckR_1\,e^{-k(x+l_1)} - ck(\operatorname{sign} x)R_0\,e^{-k|x|} + ckR_2\,e^{-k(l_2-x)} \tag{9}$$

and

$$\frac{d^2v}{dx^2} = -2ckp(x) + ck^2\int_{-l_1}^{x} e^{-k(x-\xi)}p(\xi)\,d\xi + ck^2\int_{x}^{l_2} e^{-k(\xi-x)}p(\xi)\,d\xi + ck^2R_1\,e^{-k(x\pm l_1)} + ck^2R_0\,e^{-k|x|} + ck^2R_2\,e^{-k(l_2-x)}. \tag{10}$$

From equations (7) and (10), it is found that

$$p(x) = -\frac{1}{2ck}\frac{d^2v}{dx^2} + \frac{k}{2c}v(x). \tag{11}$$

Then, from the well-known relations

$$\frac{\mathrm{d}M}{\mathrm{d}x} = Q, \qquad \frac{\mathrm{d}Q}{\mathrm{d}x} = -[q(x)-p(x)]$$

and the expression of curvature of the beam (6) is derived the differential equation for the deflexion curve of the beam

$$EI\left(1+\frac{\varkappa}{2ckGA}\right)\frac{\mathrm{d}^4v}{\mathrm{d}x^4}-\frac{1}{2ck}\left(1+\frac{\varkappa EIk^2}{GA}\right)\frac{\mathrm{d}^2v}{\mathrm{d}x^2}+\frac{k}{2c}v(x) = q-\frac{\varkappa EI}{GA}\frac{\mathrm{d}^2q}{\mathrm{d}x^2}. \tag{12}$$

For the sake of brevity, the following notations are introduced

$$b = \frac{\varkappa}{2ckGA}, \qquad d = \frac{\varkappa EIk^2}{GA}, \qquad \beta_0 = \left[\frac{k}{8cEI(1+b)}\right]^{\frac{1}{4}}$$
$$a = \frac{(1+d)/2ckEI(1+b)}{2[k/2cEI(1+b)]^{\frac{1}{2}}} \tag{13}$$

on the basis of which equation (12) acquires the form

$$\frac{\mathrm{d}^4v}{\mathrm{d}x^4}-4a\beta_0^2\frac{\mathrm{d}^2v}{\mathrm{d}x^2}+4\beta_0^4v(x) = \frac{1}{EI(1+b)}q(x)-\frac{\varkappa}{GA(1+b)}\frac{\mathrm{d}^2q}{\mathrm{d}x^2}. \tag{14}$$

Differential equation (14) is valid in all intervals where q and p are continuous functions to their derivatives of the second-order. Its general solution is the sum of the solution of the corresponding homogeneous differential equation and a particular solution.

First, there is formulated the solution of the homogeneous differential equation

$$\frac{\mathrm{d}^4v}{\mathrm{d}x^4}-4a\beta_0^2\frac{\mathrm{d}^2v}{\mathrm{d}x^2}+4\beta_0^4v(x) = 0. \tag{15}$$

The solution assumes different forms, dependent upon whether the non-negative quantity a is <1, >1 or $=1$.

(i) When $a < 1$, then

$$v(x) = B_1\,\mathrm{e}^{\alpha x}\cos\beta x+B_2\,\mathrm{e}^{\alpha x}\sin\beta x+B_3\,\mathrm{e}^{-\alpha x}\cos\beta x+B_4\,\mathrm{e}^{-\alpha x}\sin\beta x \tag{16}$$

where α and β denote

$$\alpha = \beta_0\sqrt{(1+a)}, \qquad \beta = \beta_0\sqrt{(1-a)}. \tag{17}$$

(ii) In the case $a > 1$, all the roots are real, and the solution

$$v(x) = B_1\,\mathrm{e}^{\gamma x}+B_2\,\mathrm{e}^{-\gamma x}+B_3\,\mathrm{e}^{\delta x}+B_4\,\mathrm{e}^{-\delta x} \tag{18}$$

is obtained.

Here, γ and δ denote the expressions

$$\gamma = \beta_0(\sqrt{2})\sqrt{[a+\sqrt{(a^2-1)}]}, \qquad \delta = \beta_0(\sqrt{2})\sqrt{[a-\sqrt{(a^2-1)}]}. \tag{19}$$

(iii) In the last case, $a = 1$, the solution is

$$v(x) = B_1\,\mathrm{e}^{\varepsilon x}+B_2\varepsilon x\,\mathrm{e}^{\varepsilon x}+B_3\,\mathrm{e}^{-\varepsilon x}+B_4\varepsilon x\,\mathrm{e}^{-\varepsilon x} \tag{20}$$

where ε denotes

$$\varepsilon = \beta_0\sqrt{2}. \tag{21}$$

According to the order of differential equation (15), there are four integration constants in solutions (16), (18) and (20) to be determined on the basis of the boundary conditions of the beam. Boundary conditions are specifications concerning the deflexion, the slope of the deflexion curve, the bending moment or the shear force of the beam. For this purpose, the general expressions of the bending moment and the shear force of the beam will be introduced. From equation (6) it follows, on the application of notation (13) and the expression of $p(x)$ (11), that

$$M(x) = -EI(1+b)\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}+EIk^2 bv(x)-\frac{\varkappa EI}{GA}q(x) \tag{22}$$

and differentiating (22)

$$Q(x) = -EI(1+b)\frac{\mathrm{d}^3 v}{\mathrm{d}x^3}+EIk^2 b\frac{\mathrm{d}v}{\mathrm{d}x}-\frac{\varkappa EI}{GA}\frac{\mathrm{d}q}{\mathrm{d}x}. \tag{23}$$

Further to the boundary conditions, certain conditions are needed for determination of the concentrated foundation pressures. By utilizing formula (9), at the point of application of the single load F, there is obtained

$$\left[\frac{\mathrm{d}v}{\mathrm{d}x}\right]_{x=+0}-\left[\frac{\mathrm{d}v}{\mathrm{d}x}\right]_{x=-0} = -2ckR_0 \tag{24}$$

where $[\mathrm{d}v/\mathrm{d}x]_{x=+0}$ and $[\mathrm{d}v/\mathrm{d}x]_{x=-0}$ are the values of the slope when x tends to zero from the right and from the left, respectively. On the other hand, from the expression of the curvature of the beam (6) it is found that

$$\left[\frac{\mathrm{d}v}{\mathrm{d}x}\right]_{x=+0}-\left[\frac{\mathrm{d}v}{\mathrm{d}x}\right]_{x=-0} = \frac{\varkappa}{GA}[Q(+0)-Q(-0)] \tag{25}$$

and, if the relations $Q(+0)-Q(-0) = -(F-R_0)$ and $b = \varkappa/2ckGA$ are taken into account, then

$$R_0 = \frac{b}{1+b}F. \tag{26}$$

At the right-end of the beam, on application of the first derivative of expression (8) and equation (9),

$$\left[\frac{\mathrm{d}v}{\mathrm{d}x}\right]_{x=l_2+0}-\left[\frac{\mathrm{d}v}{\mathrm{d}x}\right]_{x=l_2-0} = -2ckR_2$$

and, on comparison of $[\mathrm{d}v/\mathrm{d}x]_{x=l_2+0}$ with deflection $v(l_2)$, according to formula (7), finally,

$$\left[\frac{\mathrm{d}v}{\mathrm{d}x}\right]_{x=l_2-0}+kv(l_2) = 2ckR_2. \tag{27}$$

Similarly, at the left-end of the beam, there is obtained

$$\left[\frac{dv}{dx}\right]_{x=-l_1+0} - kv(-l_1) = -2ckR_1. \tag{28}$$

3. SPECIAL CASES OF LOADING OF BEAMS OF FINITE LENGTH

Below, there are solved two simple loading cases of beams of finite length, viz. a uniformly distributed load on the beam and a concentrated force at the middle of the beam.

3.1 *Uniformly distributed load*

The origin of co-ordinates at the middle of the beam is taken (Fig. 3). By reason of symmetry, only the positive branch of the beam is considered. First, the boundary conditions which impose the integration constants and the concentrated foundation pressure at the end of the beam will be established.

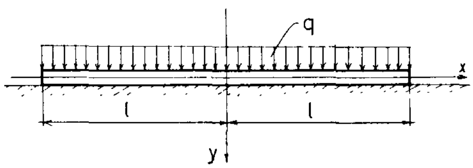

FIG. 3

It follows from symmetry that the slope of the deflexion curve and the shear force are to be zero at the origin. At the free-end of the beam, the bending moment disappears, and the shear force has the value $-R$, the concentrated foundation pressure according to condition (27). The desired conditions are accordingly

$$\left.\begin{aligned}\left[\frac{dv}{dx}\right]_{x=0} &= 0\\ Q(0) &= 0\\ M(l) &= 0\\ Q(l) &= -R\\ \left[\frac{dv}{dx}\right]_{x=l} + kv(l) &= 2ckR\end{aligned}\right\} \tag{29}$$

A particular solution of differential equation (14) is

$$v_0 = \frac{2c}{k}q \tag{30}$$

in which q is the intensity of the uniformly distributed load. The solution of the homogeneous differential equation (15) has the form (16), (18) or (20), dependent upon the value of parameter a. The procedure of determination of the integration constants is routine, hence its details are omitted, and only the final results are given.

The equations are valid for $0 \leqq x \leqq l$.

3.1.1 *Case* $a < 1$.

$$\left.\begin{aligned}
v(x) &= \frac{2c}{k}q + \frac{8R\beta_0 c(1+b)}{kN_1}\left\{\frac{d-1}{d+1}a[\cosh \alpha l \sinh \alpha x \sin \beta(l-x)\right.\\
&\quad + \sin \beta l \cos \beta x \sinh \alpha(l-x)] - \sqrt{(1-a^2)}[\cosh \alpha l \cosh \alpha x \cos \beta(l-x)\\
&\quad \left. - \sin \beta l \sin \beta x \cosh \alpha(l-x)]\right\}.\\
M(x) &= \frac{2R}{\beta_0 N_1}[\cosh \alpha l \sinh \alpha x \sin \beta(l-x) + \sin \beta l \cos \beta x \sinh \alpha(l-x)].\\
Q(x) &= -\frac{2R}{N_1}\{\sqrt{(1-a)}[\cosh \alpha l \sinh \alpha x \cos \beta(l-x) + \sin \beta l \sin \beta x \sinh \alpha(l-x)]\\
&\quad + \sqrt{(1+a)}[\sin \beta l \cos \beta x \cosh \alpha(l-x) - \cosh \alpha l \cosh \alpha x \sin \beta(l-x)]\}.\\
p(x) &= q - \frac{4R\beta_0}{N_1}\{\sqrt{(1-a^2)}[\cosh \alpha l \cosh \alpha x \cos \beta(l-x) - \sin \beta l \sin \beta x \cosh \alpha(l-x)]\\
&\quad - a[\cosh \alpha l \sinh \alpha x \sin \beta(l-x) + \sin \beta l \cos \beta x \sinh \alpha(l-x)]\}.\\
R &= q \Bigg/ k\left\{1 + \frac{2\beta_0(1+b)}{kN_1}\left[\sqrt{(1-a^2)}(\cosh 2\alpha l + \cos 2\beta l) + \right.\right.\\
&\quad \left.\left. + \frac{\beta_0}{k}\left(1 + \frac{2ad}{1+d}\right)\sqrt{(1-a)}\sinh 2\alpha l - \frac{\beta_0}{k}\left(1 - \frac{2ad}{1+d}\right)\sqrt{(1+a)}\sin 2\beta l\right]\right\}.
\end{aligned}\right\} \tag{31}$$

Factor N_1 denotes

$$N_1 = \sqrt{(1-a)}\sinh 2\alpha l + \sqrt{(1+a)}\sin 2\beta l. \tag{32}$$

3.1.2 *Case* $a > 1$.

$$\left.\begin{aligned}
v(x) &= \frac{2c}{k}q + \frac{4R\beta_0 c(1+b)}{kN_2}\left[-\left(\frac{d-1}{d+1}a + \sqrt{(a^2-1)}\right)\cosh \delta l \cosh \gamma x\right.\\
&\quad \left. + \left(\frac{d-1}{d+1}a - \sqrt{(a^2-1)}\right)\cosh \gamma l \cosh \delta x\right].\\
M(x) &= \frac{R}{\beta_0 N_2}(-\cosh \delta l \cosh \gamma x + \cosh \gamma l \cosh \delta x).\\
Q(x) &= -\frac{(\sqrt{2})R}{N_2}[\sqrt{[a+\sqrt{(a^2-1)}]}\cosh \delta l \sinh \gamma x - \sqrt{[a-\sqrt{(a^2-1)}]}\cosh \gamma l \sinh \delta x]\\
p(x) &= q - \frac{2R\beta_0}{N_2}[(a+\sqrt{(a^2-1)})\cosh \delta l \cosh \gamma x - (a-\sqrt{(a^2-1)})\cosh \gamma l \cosh \delta x].
\end{aligned}\right\} \tag{33}$$

$$\left.\begin{aligned} R = q\Big/k\Bigg\{1+\frac{2\beta_0(1+b)}{kN_2}\Bigg[&2\surd(a^2-1)\cosh\gamma l\cosh\delta l\\ &+\frac{\beta_0}{\surd 2k}\left(\frac{d-1}{d+1}a+\surd(a^2-1)\right)\surd[a+\surd(a^2-1)](\sinh(\gamma+\delta)l+\sinh(\gamma-\delta)l)\\ &-\frac{\beta_0}{\surd 2k}\left(\frac{d-1}{d+1}a-\surd(a^2-1)\right)\surd[a-\surd(a^2-1)](\sinh(\gamma+\delta)l-\sinh(\gamma-\delta)l\Bigg]\Bigg\}.\end{aligned}\right\}$$

Factor N_2 denotes

$$N_2 = \surd(a-1)\sinh(\gamma+\delta)l+\surd(a+1)\sinh(\gamma-\delta)l. \tag{34}$$

3.1.3 *Case* $a = 1$.

$$\left.\begin{aligned} v(x) &= \frac{2c}{k}q+\frac{4(\surd 2)R\beta_0c(1+b)}{kN_3}\left[\left(-2\cosh\varepsilon l+\frac{d-1}{d+1}\varepsilon l\sinh\varepsilon l\right)\cosh\varepsilon x\right.\\ &\qquad\left.-\frac{d-1}{d+1}\varepsilon x\cosh\varepsilon l\sinh\varepsilon x\right].\\ M(x) &= \frac{(\surd 2)R}{\beta_0N_3}(\varepsilon l\sinh\varepsilon l\cosh\varepsilon x-\varepsilon x\cosh\varepsilon l\sinh\varepsilon x).\\ Q(x) &= -\frac{2R}{N_3}[(\cosh\varepsilon l-\varepsilon l\sinh\varepsilon l)\sinh\varepsilon x+\varepsilon x\cosh\varepsilon l\cosh\varepsilon l].\\ p(x) &= q-\frac{2(\surd 2)R\beta_0}{N_3}[(2\cosh\varepsilon l-\varepsilon l\sinh\varepsilon l)\cosh\varepsilon x+\varepsilon x\cosh\varepsilon l\sinh\varepsilon x].\\ R &= q\Big/k\Bigg\{1+\frac{2\beta_0(1+b)}{kN_3}\Bigg[(\surd 2)(\cosh 2\varepsilon l+1)\\ &\qquad+\frac{\beta_0}{k}\left(\left(2+\frac{d-1}{d+1}\right)\sinh 2\varepsilon l+\frac{d-1}{d+1}2\varepsilon l\right)\Bigg]\Bigg\}.\end{aligned}\right\} \tag{35}$$

Factor N_3 denotes

$$N_3 = \sinh 2\varepsilon l+2\varepsilon l. \tag{36}$$

3.2 *Concentrated force at the middle of the beam*

The origin of the co-ordinates is taken at the point of application of the load F, i.e. at the middle of the beam (Fig. 4). By reason of symmetry, it follows that only the positive part of the beam needs consideration. Again, there must first be found the appropriate boundary conditions for determination of the integration constants, and concentrated foundation pressures R_0 at the point of application of load F and R at the end of the beam.

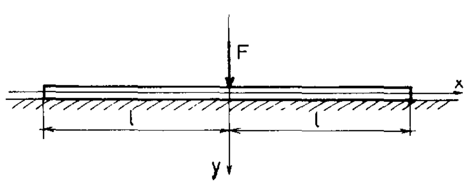

FIG. 4

With the occurring symmetry taken into account and by making use of formula (25), there are obtained

$$
\begin{aligned}
\left[\frac{\mathrm{d}v}{\mathrm{d}x}\right]_{x=0} &= -\frac{\varkappa(F-R_0)}{2GA} \\
Q(0) &= -\frac{1}{2}(F-R_0).
\end{aligned}
\tag{37'}
$$

At the free end of the beam, it readily follows that

$$
\begin{aligned}
M(l) &= 0 \\
Q(l) &= -R.
\end{aligned}
\tag{37''}
$$

Foundation pressures R_0 and R are defined on the basis of formulae (26) and (27) by

$$
\begin{aligned}
R_0 &= \frac{b}{1+b}F \\
\left[\frac{\mathrm{d}v}{\mathrm{d}x}\right]_{x=l} + kv(l) &= 2ckR
\end{aligned}
\tag{37'''}
$$

The integration constants B_1, B_2, B_3 and B_4 in expressions (16), (18) or (20) will be determined from conditions (37') and (37''). If there is taken into account the value of $R_0 = Fb/(1+b)$, these constants are still dependent upon foundation pressure R at the end of the beam. The value of R follows from condition (37'''). On proceeding in this way, and omitting the uninteresting computations, the following results are obtained.

3.2.1 *Case $a < 1$.*

$$
\begin{aligned}
v(x) = \frac{F\beta_0 c}{kN_1}\Bigg\{&\left(1+\frac{2ad}{1+a}\right)\sqrt{\left(\frac{1-a}{1+a}\right)}[\cosh \alpha l \cosh \alpha(l-x) + \sinh \alpha l \sinh \alpha(l-x)] \cos \beta x \\
&+\left(1-\frac{2ad}{1+d}\right)[\cosh \alpha l \sinh \alpha(l-x) + \sinh \alpha l \cosh \alpha(l-x)] \sin \beta x \\
&+\left(1-\frac{2ad}{1+d}\right)\sqrt{\left(\frac{1+a}{1-a}\right)}[\cos \beta l \cos \beta(l-x) - \sin \beta l \sin \beta(l-x)] \cosh \alpha x \\
&-\left(1+\frac{2ad}{1+d}\right)[\cos \beta l \sin \beta(l-x) + \sin \beta l \cos \beta(l-x)] \sinh \alpha x \\
&+\frac{2}{\sqrt{(1-a^2)}}\left(1-\frac{2a^2}{1+d}\right) \cosh \alpha x \cos \beta x - \frac{4a}{1+d} \sinh \alpha x \sin \beta x \\
&+8\left(\frac{R}{F}\right)(1+b)\Bigg[\left(\frac{d-1}{d+1}a \sinh \alpha l \sin \beta l - \sqrt{(1-a^2)}\cosh \alpha l \cos \beta l\right) \cosh \alpha x \cos \beta x \\
&-\left(\frac{d-1}{d+1}a \cosh \alpha l \cos \beta l + \sqrt{(1-a^2)} \sinh \alpha l \sin \beta l\right) \sinh \alpha x \sin \beta x\Bigg]\Bigg\}.
\end{aligned}
$$

$$
\left.
\begin{aligned}
M(x) = {} & \frac{F}{4\beta_0(1+b)N_1}\Bigg\{\sqrt{\left(\frac{1-a}{1+a}\right)}[\cosh\alpha l\cosh\alpha(l-x)+\sinh\alpha l\sinh\alpha(l-x)]\cos\beta x \\
& -[\cosh\alpha l\sinh\alpha(l-x)+\sinh\alpha l\cosh\alpha(l-x)]\sin\beta x \\
& -\sqrt{\left(\frac{1+a}{1-a}\right)}[\cos\beta l\cos\beta(l-x)-\sin\beta l\sin\beta(l-x)]\cosh\alpha x \\
& -[\cos\beta l\sin\beta(l-x)+\sin\beta l\cos\beta(l-x)]\sinh\alpha x \\
& +\frac{2a}{\sqrt{(1-a^2)}}\cosh\alpha x\cos\beta x+2\sinh\alpha x\sin\beta x \\
& +8\left(\frac{R}{F}\right)(1+b)[\cosh\alpha l\sin\beta(l-x)\sinh\alpha x+\sin\beta l\sinh\alpha(l-x)\cos\beta x]\Bigg\}. \\
Q(x) = {} & -\frac{F}{2(1+b)N_1}\Bigg\{-\frac{a}{\sqrt{(1+a)}}[\cosh\alpha l\cosh\alpha(l-x)+\sinh\alpha l\sinh\alpha(l-x)]\sin\beta x \\
& +\sqrt{(1-a)}[\cosh\alpha l\sinh\alpha(l-x)+\sinh\alpha l\cosh\alpha(l-x)]\cos\beta x \\
& +\frac{a}{\sqrt{(1-a)}}[\cos\beta l\cos\beta(l-x)-\sin\beta l\sin\beta(l-x)]\sinh\alpha x \\
& +\sqrt{(1+a)}[\cos\beta l\sin\beta(l-x)+\sin\beta l\cos\beta(l-x)]\cosh\alpha x \\
& -\frac{1}{\sqrt{(1-a)}}\sinh\alpha x\cos\beta x-\frac{1}{\sqrt{(1+a)}}\cosh\alpha x\sin\beta x \\
& +4\left(\frac{R}{F}\right)(1+b)[(\sqrt{(1-a)}\cosh\alpha l\cos\beta l-\sqrt{(1+a)}\sinh\alpha l\sin\beta l)\sinh\alpha x\cos\beta x \\
& +(\sqrt{(1+a)}\cosh\alpha l\cos\beta l+\sqrt{(1-a)}\sinh\alpha l\sin\beta l)\cosh\alpha x\sin\beta x]\Bigg\}. \\
p(x) = {} & \frac{F\beta_0}{2(1+b)N_1}\Bigg\{(1+2a)\sqrt{\left(\frac{1-a}{1+a}\right)}[\cosh\alpha l\cosh\alpha(l-x)+\sinh\alpha l\sinh\alpha(l-x)]\cos\beta x \\
& +(1-2a)[\cosh\alpha l\sinh\alpha(l-x)+\sinh\alpha l\cosh\alpha(l-x)]\sin\beta x \\
& +(1-2a)\sqrt{\left(\frac{1+a}{1-a}\right)}[\cos\beta l\cos\beta(l-x)-\sin\beta l\sin\beta(l-x)]\cosh\alpha x \\
& -(1+2a)[\cos\beta l\sin\beta(l-x)+\sin\beta l\cos\beta(l-x)]\sinh\alpha x \\
& +\frac{2}{\sqrt{(1-a^2)}}\cosh\alpha x\cos\beta x \\
& +8\left(\frac{R}{F}\right)(1+b)[(a\sinh\alpha l\sin\beta l-\sqrt{(1-a^2)}\cosh\alpha l\cos\beta l)\cosh\alpha x\cos\beta x \\
& -(a\cosh\alpha l\cos\beta l+\sqrt{(1-a^2)}\sinh\alpha l\sin\beta l)\sinh\alpha x\sin\beta x]\Bigg\}. \\
R = {} & \frac{2F\beta_0}{kN_1}\left(H_1+\frac{\beta_0}{k}H_3\right)\left[1+\frac{2\beta_0(1+b)}{kN_1}\left(H_2+\frac{\beta_0}{k}H_4\right)\right]^{-1}.
\end{aligned}
\right\}
\quad (38)
$$

The abbreviations are

$$H_1 = \sqrt{(1-a^2)}\cosh \alpha l \cos \beta l - a \sinh \alpha l \sin \beta l.$$

$$H_2 = \sqrt{(1-a^2)}(\cosh 2\alpha l + \cos 2\beta l).$$

$$H_3 = \sqrt{(1-a)}\sinh \alpha l \cos \beta l - \sqrt{(1+a)}\cosh \alpha l \sin \beta l.$$

$$H_4 = \left(1+\frac{2ad}{1+d}\right)\sqrt{(1-a)}\sinh 2\alpha l - \left(1-\frac{2ad}{1+d}\right)\sqrt{(1+a)}\sin 2\beta l.$$

3.2.2 *Case $a > 1$.*

$$\left.\begin{aligned}
v(x) = {} & \frac{F\beta_0 c}{(\sqrt{2})kN_2}\left\{\left(\frac{d-1}{d+1}a+\sqrt{(a^2-1)}\right)\left[-\left(\frac{2a-1}{\sqrt{(a-1)}}-\frac{2a+1}{\sqrt{(a+1)}}\right)\sqrt{[a+\sqrt{(a^2-1)}]}\cosh \gamma x\right.\right. \\
& +\left(\frac{2a-1}{\sqrt{(a-1)}}+\frac{2a+1}{\sqrt{(a+1)}}\right)\sqrt{[a-\sqrt{(a^2-1)}]}\cosh \delta l \cosh \gamma(l-x) \\
& \left.-\left(\frac{2a-1}{\sqrt{(a-1)}}+\frac{2a+1}{\sqrt{(a+1)}}\right)(\sqrt{[a-\sqrt{(a^2-1)}]})^3 \sinh \delta l \sinh \gamma(l-x)\right] \\
& +\left(\frac{d-1}{d+1}a-\sqrt{(a^2-1)}\right)\left[-\left(\frac{2a-1}{\sqrt{(a-1)}}+\frac{2a+1}{\sqrt{(a+1)}}\right)\sqrt{[a-\sqrt{(a^2-1)}]}\cosh \delta x\right. \\
& +\left(\frac{2a-1}{\sqrt{(a-1)}}-\frac{2a+1}{\sqrt{(a+1)}}\right)\sqrt{[a+\sqrt{(a^2-1)}]}\cosh \gamma l \cosh \delta(l-x) \\
& \left.-\left(\frac{2a-1}{\sqrt{(a-1)}}-\frac{2a+1}{\sqrt{(a+1)}}\right)(\sqrt{[a+\sqrt{(a^2-1)}]})^3 \sin \gamma l \sinh \delta(l-x)\right] \\
& +4(\sqrt{2})\left(\frac{R}{F}\right)(1+b)\left[-\left(\frac{d-1}{d+1}a+\sqrt{(a^2-1)}\right)\cosh \delta l \cosh \gamma x\right. \\
& \left.\left.+\left(\frac{d-1}{d+1}a-\sqrt{(a^2-1)}\right)\cosh \gamma l \cosh \delta x\right]\right\}. \\
M(x) = {} & \frac{F}{4(\sqrt{2})\beta_0(1+b)N_2}\left[-\left(\frac{2a-1}{\sqrt{(a-1)}}-\frac{2a+1}{\sqrt{(a+1)}}\right)\sqrt{[a+\sqrt{(a^2-1)}]}\cosh \gamma x\right. \\
& +\left(\frac{2a-1}{\sqrt{(a-1)}}+\frac{2a+1}{\sqrt{(a+1)}}\right)\sqrt{[a-\sqrt{(a^2-1)}]}\cosh \delta l \cosh \gamma(l-x) \\
& -\left(\frac{2a-1}{\sqrt{(a-1)}}+\frac{2a+1}{\sqrt{(a+1)}}\right)(\sqrt{[a-\sqrt{(a^2-1)}]})^3 \sinh \delta l \sinh \gamma(l-x) \\
& -\left(\frac{2a-1}{\sqrt{(a-1)}}+\frac{2a+1}{\sqrt{(a+1)}}\right)\sqrt{[a-\sqrt{(a^2-1)}]}\cosh \delta x \\
& +\left(\frac{2a-1}{\sqrt{(a-1)}}-\frac{2a+1}{\sqrt{(a+1)}}\right)\sqrt{[a+\sqrt{(a^2-1)}]}\cosh \gamma l \cosh \delta(l-x) \\
& -\left(\frac{2a-1}{\sqrt{(a-1)}}-\frac{2a+1}{\sqrt{(a+1)}}\right)(\sqrt{[a+\sqrt{(a^2-1)}]})^3 \sinh \gamma l \sinh \delta(l-x) \\
& \left.+4(\sqrt{2})\left(\frac{R}{F}\right)(1+b)(-\cosh \delta l \cosh \gamma x + \cosh \gamma l \cosh \delta x)\right].
\end{aligned}\right\} \quad (39)$$

$$
\begin{aligned}
Q(x) = {} & -\frac{F}{4(1+b)N_2}\Bigg[\left(\frac{2a-1}{\surd(a-1)}-\frac{2a+1}{\surd(a+1)}\right)[a+\surd(a^2-1)]\sinh \gamma x \\
& +\left(\frac{2a-1}{\surd(a-1)}+\frac{2a+1}{\surd(a+1)}\right)\cosh \delta l \sinh \gamma(l-x) \\
& -\left(\frac{2a-1}{\surd(a-1)}+\frac{2a+1}{\surd(a+1)}\right)[a-\surd(a^2-1)]\sinh \delta l \cosh \gamma(l-x) \\
& +\left(\frac{2a-1}{\surd(a-1)}+\frac{2a+1}{\surd(a+1)}\right)[a-\surd(a^2-1)]\sinh \delta x \\
& +\left(\frac{2a-1}{\surd(a-1)}-\frac{2a+1}{\surd(a+1)}\right)\cosh \gamma l \sinh \delta(l-x) \\
& -\left(\frac{2a-1}{\surd(a-1)}-\frac{2a+1}{\surd(a+1)}\right)[(a+\surd(a^2-1)]\sinh \gamma l \cosh \delta(l-x) \\
& +4(\surd 2)\left(\frac{R}{F}\right)(1+b)(\surd[a+\surd(a^2-1)]\cosh \delta l \sinh \gamma x \\
& -\surd[a-\surd(a^2-1)]\cosh \gamma l \sinh \delta x)\Bigg].
\end{aligned}
$$

$$
\begin{aligned}
p(x) = {} & \frac{F\beta_0}{2(\surd 2)(1+b)N_2}\Bigg\{-\left(\frac{2a-1}{\surd(a-1)}-\frac{2a+1}{\surd(a+1)}\right)(\surd[a+\surd(a^2-1)])^3\cosh \gamma x \\
& +\left(\frac{2a-1}{\surd(a-1)}+\frac{2a+1}{\surd(a+1)}\right)\surd[a+\surd(a^2-1)]\cosh \delta l \cosh \gamma(l-x) \\
& -\left(\frac{2a-1}{\surd(a-1)}+\frac{2a+1}{\surd(a+1)}\right)\surd[a-\surd(a^2-1)]\sinh \delta l \sinh \gamma(l-x) \\
& -\left(\frac{2a-1}{\surd(a-1)}+\frac{2a+1}{\surd(a+1)}\right)(\surd[a-\surd(a^2-1)])^3\cosh \delta x \\
& +\left(\frac{2a-1}{\surd(a-1)}-\frac{2a+1}{\surd(a+1)}\right)\surd[a-\surd(a^2-1)]\cosh \gamma l \cosh \delta(l-x) \\
& -\left(\frac{2a-1}{\surd(a-1)}-\frac{2a+1}{\surd(a+1)}\right)\surd[a+\surd(a^2-1)]\sinh \gamma l \sinh \delta(l-x) \\
& +4(\surd 2)\left(\frac{R}{F}\right)(1+b)[-(a+\surd(a^2-1))\cosh \delta l \cosh \gamma x \\
& +(a-\surd(a^2-1))\cosh \gamma l \cosh \delta x]\Bigg\}.
\end{aligned}
$$

$$
R = \frac{F\beta_0\surd(a^2-1)}{2kN_2}\left((\surd 2)H_1+2\frac{\beta_0}{k}H_3\right)\left[1+\frac{2\beta_0(1+b)}{kN_2}\left(H_2+\frac{\beta_0}{(\surd 2)k}H_4\right)\right]^{-1}.
$$

Here the abbreviations are

$$
H_1 = -\left(\frac{2a-1}{\surd(a-1)}-\frac{2a+1}{\surd(a+1)}\right)\surd[a+\surd(a^2-1)]\cosh \gamma l
$$

$$+\left(\frac{2a-1}{\sqrt{(a-1)}}+\frac{2a+1}{\sqrt{(a+1)}}\right)\sqrt{[a-\sqrt{(a^2-1)}]}\cosh\delta l$$

$$H_2 = 2\sqrt{(a^2-1)}\cosh\gamma l\cosh\delta l$$

$$H_3 = -\left(\frac{2a-1}{\sqrt{(a-1)}}-\frac{2a+1}{\sqrt{(a+1)}}\right)[a+\sqrt{(a^2-1)}]\sinh\gamma l$$

$$+\left(\frac{2a-1}{\sqrt{(a-1)}}+\frac{2a+1}{\sqrt{(a+1)}}\right)[a-\sqrt{(a^2-1)}]\sinh\delta l$$

$$H_4 = \left(\frac{d-1}{d+1}a+\sqrt{(a^2-1)}\right)\sqrt{[a+\sqrt{(a^2-1)}]}[\sinh(\gamma+\delta)l+\sinh(\gamma-\delta)l]$$

$$-\left(\frac{d-1}{d+1}a-\sqrt{(a^2-1)}\right)\sqrt{[a-\sqrt{(a^2-1)}]}[\sinh(\gamma+\delta)l-\sinh(\gamma-\delta)l].$$

3.2.3 *Case* $a = 1$.

$$\left.\begin{aligned}
v(x) = {} & \frac{F\beta_0 c}{(\sqrt{2})kN_3}\Bigg\{\left(1+\frac{2d}{1+d}\right)[\cosh\varepsilon l\cosh\varepsilon(l-x)+\sinh\varepsilon l\sinh\varepsilon(l-x) \\
& -2\varepsilon l\sinh\varepsilon x-\cosh\varepsilon x]+\left(2\varepsilon^2 l^2\frac{d-1}{d+1}+8\right)\cosh\varepsilon x \\
& +\frac{d-1}{d+1}\varepsilon x[3\sinh\varepsilon x-2\varepsilon l\cosh\varepsilon x-\sinh\varepsilon l\cosh\varepsilon(l-x)-\cosh\varepsilon l\sinh\varepsilon(l-x)] \\
& -8\left(\frac{R}{F}\right)(1+b)\bigg(2\cosh\varepsilon l\cosh\varepsilon x-\varepsilon l\frac{d-1}{d+1}\sinh\varepsilon l\cosh\varepsilon x \\
& +\frac{d-1}{d+1}\varepsilon x\cosh\varepsilon l\sinh\varepsilon x\bigg)\Bigg\}. \\
M(x) = {} & \frac{F}{4(\sqrt{2})\beta_0(1+b)N_3}\Big\{\cosh\varepsilon l\cosh\varepsilon(l-x)+\sinh\varepsilon l\sinh\varepsilon(l-x) \\
& +(2\varepsilon^2 l^2-1)\cosh\varepsilon x-2\varepsilon l\sinh\varepsilon x \\
& +\varepsilon x[3\sinh\varepsilon x-2\varepsilon l\cosh\varepsilon x-\sinh\varepsilon l\cosh\varepsilon(l-x)-\cosh\varepsilon l\sinh\varepsilon(l-x)] \\
& +8\left(\frac{R}{F}\right)(1+b)(\varepsilon l\sinh\varepsilon l\cosh\varepsilon x-\varepsilon x\cosh\varepsilon l\sinh\varepsilon x)\Big\}. \\
Q(x) = {} & -\frac{F}{4(1+b)N_3}\Big\{2\cosh\varepsilon l\sinh\varepsilon(l-x)+2\sinh\varepsilon l\cosh\varepsilon(l-x) \\
& -2(\varepsilon^2 l^2+1)\sinh\varepsilon x+4\varepsilon l\cosh\varepsilon x \\
& -\varepsilon x[3\cosh\varepsilon x-2\varepsilon l\sinh\varepsilon x+\sinh\varepsilon l\sinh\varepsilon(l-x)+\cosh\varepsilon l\cosh\varepsilon(l-x)] \\
& -8\left(\frac{R}{F}\right)(1+b)(\varepsilon l\sinh\varepsilon l\sinh\varepsilon x-\cosh\varepsilon l\sinh\varepsilon x-\varepsilon x\cosh\varepsilon l\cosh\varepsilon x)\Big\}.
\end{aligned}\right\}\quad(40)$$

$$p(x) = \frac{F\beta_0}{2(\sqrt{2})(1+b)N_3}\Bigg\{3[\cosh \varepsilon l \cosh \varepsilon(l-x) + \sinh \varepsilon l \sinh \varepsilon(l-x) - 2\varepsilon l \sinh \varepsilon x]$$
$$+(2\varepsilon^2 l^2+5)\cosh \varepsilon x + \varepsilon x[3\sinh \varepsilon x - 2\varepsilon l \cosh \varepsilon x$$
$$-\sinh \varepsilon l \cosh \varepsilon(l-x) - \cosh \varepsilon l \sinh \varepsilon(l-x)]$$
$$-8\left(\frac{R}{F}\right)(1+b)(2\cosh \varepsilon l \cosh \varepsilon x - \varepsilon l \sinh \varepsilon l \cosh \varepsilon x + \varepsilon x \cosh \varepsilon l \sinh \varepsilon x)\Bigg\}.$$
$$R = \frac{(\sqrt{2})F\beta_0}{kN_3}\left(H_1 + \frac{(\sqrt{2})\beta_0}{k}H_3\right)\left[1 + \frac{2(\sqrt{2})\beta_0(1+b)}{kN_3}\left(H_2 + \frac{\beta_0}{(\sqrt{2})k}H_4\right)\right]^{-1}.$$

The abbreviations denote the following

$$H_1 = 2\cosh \varepsilon l - \varepsilon l \sinh \varepsilon l$$
$$H_2 = 2\cosh^2 \varepsilon l$$
$$H_3 = \sinh \varepsilon l - \varepsilon l \cosh \varepsilon l$$
$$H_4 = \left(1 + \frac{2d}{1+d}\right)\sinh 2\varepsilon l + 2\varepsilon l \frac{d-1}{d+1}.$$

4. THE SOLUTION OF AN INFINITELY LONG BEAM

The solutions of an infinitely long beam corresponding to the loading cases of the finite-beam treated above can of course be established in exactly the same way, i.e. by determining the integration constants on the basis of boundary conditions. It is also possible to deduce the solutions direct from the corresponding expressions for the finite beam, letting the length of beam $2l$ in these increase to infinity. The results given here were derived in this way.

The first example, a uniformly distributed load q on the beam, is a simple one. On physical grounds it is deduced that deflection v and foundation pressure p are constants

$$v = \frac{2c}{k}q, \qquad p = q.$$

The second example is of a beam loaded by a single force F. Cases $a < 1$, > 1 and $= 1$ are considered separately.

4.1 *The case $a < 1$.* Factor N_1 (32) is written in the form

$$N_1 = \frac{1}{2}\sqrt{(1-a)}\,e^{2\alpha l}(1+[e^{-2\alpha l}])$$

where the expression in square brackets has a common factor $e^{-2\alpha l}$. First of all, it is observable from the formula of R in equations (38) that the value of concentrated foundation pressure R tends to zero as the length of the beam $(2l)$ grows without limit. From formulae

(38) there are readily obtained

$$\left.\begin{aligned}
v(x) &= \frac{F\beta_0 c}{k}\mathrm{e}^{-\alpha x}\left(\frac{1+\dfrac{2ad}{1+d}}{\sqrt{(1+a)}}\cos\beta x+\frac{1-\dfrac{2ad}{1+d}}{\sqrt{(1-a)}}\sin\beta x\right)\\
M(x) &= \frac{F}{4\beta_0(1+b)}\mathrm{e}^{-\alpha x}\left(\frac{1}{\sqrt{(1+a)}}\cos\beta x-\frac{1}{\sqrt{(1-a)}}\sin\beta x\right)\\
Q(x) &= -\frac{F}{2(1+b)}\mathrm{e}^{-\alpha x}\left(\cos\beta x-\frac{a}{\sqrt{(1-a^2)}}\sin\beta x\right)\\
p(x) &= \frac{F\beta_0}{2(1+b)}\mathrm{e}^{-\alpha x}\left(\frac{1+2a}{\sqrt{(1+a)}}\cos\beta x+\frac{1-2a}{\sqrt{(1-a)}}\sin\beta x\right)
\end{aligned}\right\} \tag{41}$$

4.2 *The case* $a > 1$. There are derived, proceeding as above,

$$\left.\begin{aligned}
v(x) = {}& \frac{F\beta_0 c}{2k}\left[\left(\frac{d-1}{d+1}a+\sqrt{(a^2-1)}\right)[a-\sqrt{(a^2-1)}]\left(\frac{2a-1}{\sqrt{(a-1)}}+\frac{2a-1}{\sqrt{(a+1)}}\right)\mathrm{e}^{-\gamma x}\right.\\
&\left.-\left(\frac{d-1}{d+1}a-\sqrt{(a^2-1)}\right)[a+\sqrt{(a^2-1)}]\left(\frac{2a-1}{\sqrt{(a-1)}}-\frac{2a+1}{\sqrt{(a+1)}}\right)\mathrm{e}^{-\delta x}\right]\\
M(x) = {}& \frac{F}{8\beta_0(1+b)}\left[[a-\sqrt{(a^2-1)}]\left(\frac{2a-1}{\sqrt{(a-1)}}+\frac{2a+1}{\sqrt{(a+1)}}\right)\mathrm{e}^{-\gamma x}\right.\\
&\left.-[a+\sqrt{(a^2-1)}]\left(\frac{2a-1}{\sqrt{(a-1)}}-\frac{2a+1}{\sqrt{(a+1)}}\right)\mathrm{e}^{-\delta x}\right]\\
Q(x) = {}& -\frac{F}{4(\sqrt{2})(1+b)}\left[\sqrt{[a-\sqrt{(a^2-1)}]}\left(\frac{2a-1}{\sqrt{(a-1)}}+\frac{2a+1}{\sqrt{(a+1)}}\right)\mathrm{e}^{-\gamma x}\right.\\
&\left.-\sqrt{[a+\sqrt{(a^2-1)}]}\left(\frac{2a-1}{\sqrt{(a-1)}}-\frac{2a+1}{\sqrt{(a+1)}}\right)\mathrm{e}^{-\delta x}\right]\\
p(x) = {}& \frac{F\beta_0}{4(1+b)}\left[\left(\frac{2a-1}{\sqrt{(a-1)}}+\frac{2a+1}{\sqrt{(a+1)}}\right)\mathrm{e}^{-\gamma x}-\left(\frac{2a-1}{\sqrt{(a-1)}}-\frac{2a+1}{\sqrt{(a+1)}}\right)\mathrm{e}^{-\delta x}\right].
\end{aligned}\right\} \tag{42}$$

4.3 *Finally, in case* $a = 1$

$$\left.\begin{aligned}
v(x) &= \frac{F\beta_0 c}{(\sqrt{2})k}\mathrm{e}^{-\varepsilon x}\left(1+\frac{2d}{1+d}-\frac{d-1}{d+1}\varepsilon x\right)\\
M(x) &= \frac{F}{4(\sqrt{2})\beta_0(1+b)}\mathrm{e}^{-\varepsilon x}(1-\varepsilon x)\\
Q(x) &= -\frac{F}{4(1+b)}\mathrm{e}^{-\varepsilon x}(2-\varepsilon x)\\
p(x) &= \frac{F\beta_0}{2(\sqrt{2})(1+b)}\mathrm{e}^{-\varepsilon x}(3-\varepsilon x).
\end{aligned}\right\} \tag{43}$$

5. CONCLUDING REMARKS

Results obtained by Wieghardt for an infinitely-rigid beam follow as limiting cases from formulae (31) and (38).

As mentioned before, the corresponding Winkler-type foundation is arrived at if the ratio $k/2c$ is fixed and $k \to \infty$. In this case formulae (31), (33), (35), (38). . . . , (43) yield the results obtained previously [7–9] for Winkler-type foundation with the modulus of foundation $C = k/2c$.

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Résumé—Il est présumé que l'équation differentielle de la courbe de déflection de la poutre (6) dans laquelle l'effet de cisaillement sur la courbure de la poutre a été pris en considération, est valide pour une poutre sur une fondation élastique. Selon Wieghardt, il y a relations présumées (4) avec la fonction du noyau (5) entre la déflection et la pression de fondation. Sur base de cas assomptions l'équation differentielle de courbe de déflection (14) est alors obtenue. Cette équation differentielle a été employée en examinant le cas de chargement d'une poutre de longueur finie. Les résultats correspondants pour une poutre infiniment longue sont également dérivés.

Zusammenfassung—Es wird vorausgesetzt, dass die Differentialgleichung der Balken-Durchbiegungskurve (6) worin der Einfluss der Schubspannung auf die Krümmung berücksichtigt wurde, für einen Balken auf elastischer Unterlage gültig ist. Nach Wieghardt wird ein Zusammenhang (4) mit der Kernfunktion (5) zwischen Durchbiegung und Bettungsdruck vorausgesetzt. Diesen Voraussetzungen entsprechend wird dann die Differentialgleichung der Durchbiegekurve (14) erhalten. Diese Gleichung wird angewandt zur Untersuchung zweier Belastungsfälle eines Balkens mit endlicher Länge. Die entsprechenden Resultate für unendlich lange Balken werden auch abgeleitet.

Абстракт—Предполагается, что дифференциальное уравнение кривой отклонения балки (6), в котором принимается во внимание эффект сдвига на кривизну балки правильно для балки на эластичном основании. По *Виехарту* (Wieghardt) предполагается взаимоотношение (4) керн-функции (5) между отклонением и давлением основания. На основании этих предположений получено дифференциальное уравнение кривой отклонения (14). Это дифференциальное уравнение употребляется при исследовании двух нагрузочных камер балки конечной длины. Выведены также соответствующие результаты для бесконечно длинной балки.